HORNSBY GIRLS HIGH SCHOOL

Mathematics Extension 2

Year 12 Higher School Certificate
Trial Examination Term 3 2024

STUDENT NUMBER:__________________ **STUDENT NAME:**_______________________ **TEACHER:**______________

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- NESA-approved calculators may be used
- A reference sheet is provided separately
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for untidy and poorly arranged work
- Do not use correction fluid or tape
- Do not remove this paper from the examination room

Total marks – 100

Section I Pages 3 – 7

10 marks

Attempt Questions 1 – 10

Answer on the Objective Response Answer Sheet provided

Section II Pages 8 – 14

90 marks

Attempt Questions 11 – 16

Start each question in a new writing booklet

Write your student number on every writing booklet

Question	**1-10**	**11**	**12**	**13**	**14**	**15**	**16**	***Total***
Total	*/10*	*/15*	*/15*	*/15*	*/15*	*/15*	*/15*	*/100*

This assessment task constitutes 30% of the Higher School Certificate Course School Assessment

Outcomes assessed: MEX12-1, MEX12-2, MEX12-3, MEX12-4, MEX12-7, MEX12-8, MEX12-9.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section.

Use the Objective Response answer sheet for Questions 1 – 10

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1 Which of the following is the correct expression for $\int \frac{-1}{\sqrt{1-16x^2}}dx$?

(A) $\frac{1}{4}\cos^{-1}\frac{x}{4}+C$

(B) $4\cos^{-1}\frac{x}{4}+C$

(C) $\frac{1}{4}\cos^{-1}4x+C$

(D) $4\cos^{-1}4x+C$

2. If $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are the position vectors of the three vertices of a triangle, and $\left(\underset{\sim}{b}-\underset{\sim}{a}\right)=\lambda\left(\underset{\sim}{c}-\underset{\sim}{a}\right)$ where $0<\lambda<1$ which of the following must be false?

(A) Point *B* is diving the interval *AC* in the ratio of $\lambda:(1-\lambda)$ internally.

(B) Point *B* is diving the interval *AC* in the ratio of $\lambda:(1-\lambda)$ externally.

(C) Points *A*, *B* and *C* are collinear points.

(D) $\overrightarrow{AB}\,||\,\overrightarrow{AC}$.

3 Which of the following statements is true?

(A) $\forall a,b\in\mathbb{R}\quad \sin a=\sin b\Rightarrow a=b$

(B) $\forall a,b\in\mathbb{R}\quad |a+b|>|a-b|$

(C) $\exists a,b\in\mathbb{R}$ such that $\log_e(a+b)=\log_e(ab)$

(D) $\exists a,b\in\mathbb{C}\quad |a+b|>|a|+|b|$

4. Which of the following is the correct expression for $\int x \tan^{-1} x \, dx$

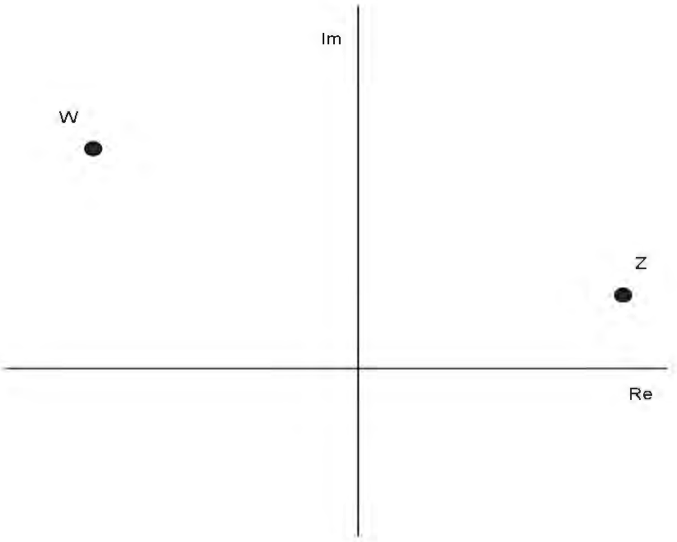
(A) $x + \tan^{-1} x + C$

(B) $\dfrac{x}{\tan^{-1} x} + \dfrac{1}{2}\ln\left(1+x^2\right) + C$

(C) $x \tan^{-1} x + \dfrac{1}{2}\ln\left(1+x^2\right) + C$

(D) $\dfrac{x^2}{2}\tan^{-1} x - \dfrac{1}{2}x + \dfrac{1}{2}\tan^{-1} x + C$.

5 The diagram below shows the position of points *z* and *w*, that have position vectors *z* and *w* respectively. In which quadrant would the point with position vector $\vec{z} - iw$ lie?



(A) Quadrant 1

(B) Quadrant 2

(C) Quadrant 3

(D) Quadrant 4

6. Consider the statement:

'For any function $f(x)$, $f(x)$ is not continuous at $x = a \Rightarrow$ $f(x)$ is not differentiable at $x = a$.'

Which of the following statement is correct?

(A) The converse statement is false and the contrapositive statement is false.

(B) The converse statement is false and the contrapositive statement is true.

(C) The converse statement is true and the contrapositive statement is false.

(D) The converse statement is true and the contrapositive statement is true.

7 Consider the vectors $\overrightarrow{OA} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB}$ with $\left|\overrightarrow{OB}\right| = 3$.

Given that $\overrightarrow{OA}.\overrightarrow{OB} = 6$, find in square units, the area of ΔOAB.

(A) $3\sqrt{10}$

(B) $\frac{3\sqrt{14}}{2}$

(C) $\frac{3\sqrt{10}}{4}$

(D) $\frac{3\sqrt{10}}{2}$

8. If $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\, dx = \frac{\pi}{2}$, what is the value of $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx$?

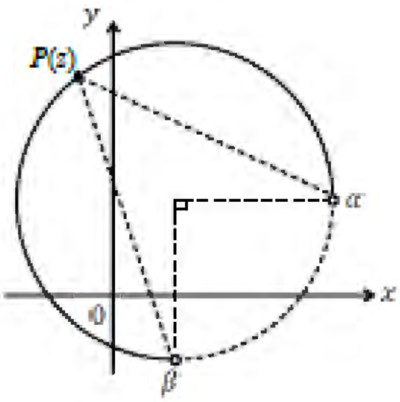
(A) 0

(B) $\frac{\pi}{2}$

(C) $\frac{\pi^2}{2}$

(D) $\frac{\pi^2}{4}$

9 The diagram shows the solution of an equation as traced out by the point $P(z)$. The path traced out by the point P representing the complex number z is three-quarter circle.



Which of the following could be the equation?

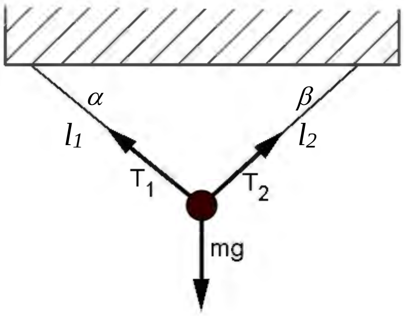
(A) $Arg(z-\alpha) - Arg(z-\beta) = 0$

(B) $Arg(z-\alpha) - Arg(z-\beta) = \frac{\pi}{4}$

(C) $Arg(z-\alpha) - Arg(z-\beta) = \frac{\pi}{2}$

(D) $Arg(z-\beta) - Arg(z-\alpha) = \frac{\pi}{4}$

10. The mass m kg is suspended from the ceiling by two light strings of equal length, where the tensions in each string are equal.



If the lengths of the strings are changed such that $l_1 > l_2$, and the angles made by the strings to the horizontal are α and β respectively, which of the following will now be true?

(A) $\alpha > \beta,\ T_1 > T_2$

(B) $\alpha > \beta,\ T_2 > T_1$

(C) $\alpha < \beta,\ T_2 > T_1$

(D) $\alpha < \beta,\ T_1 > T_2$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in a new writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a new writing booklet.

(a) Let $w=8-2i$ and $z=-5+3i$. Find $w+\bar{z}$ **1**

(b) (i) Show that $(1-2i)^2=-3-4i$ **1**

(ii) Hence solve the equation $z^2-5z+(7+i)=0$ **3**

(c) (i) Find $\frac{d}{dx}(x\sin^{-1}x)$ **1**

(ii) Hence or otherwise find $\int \sin^{-1}(x)dx$ **2**

(d) Provide a non-inductive proof as to why $12^n>5^n+7^n$, $\forall$ integers $n\geq 2$. **2**

(e) (i) Use the result $e^{in\theta}=\cos n\theta+i\sin n\theta$ to show $e^{ni\theta}+e^{-ni\theta}=2\cos n\theta$ **1**

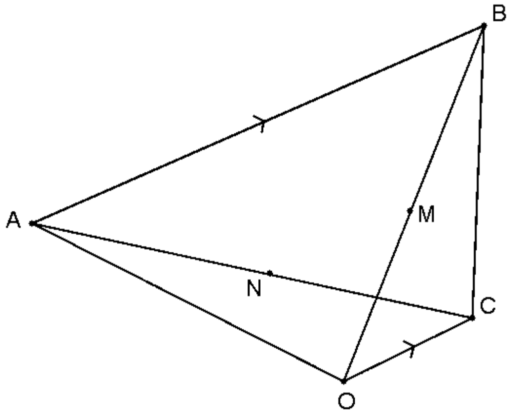
(ii) By expanding $(e^{i\theta}+e^{-i\theta})^4$, prove that $\cos^4\theta=\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}$ **2**

(iii) Hence find $\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta$ **2**

End of Question 11.

QUESTION 12 (15 marks)

(a) $OABC$ is a trapezium with $\overrightarrow{AB} = k\overrightarrow{OC}$.
Let $\underset{\sim}{a} = \overrightarrow{OA}$ and $\underset{\sim}{c} = \overrightarrow{OC}$.
M and N are midpoints of OB and AC respectively.



(i) Find $\overrightarrow{OM}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$. **2**

(ii) Find $\overrightarrow{ON}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$. **2**

(iii) Hence find $\vec{MN}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$. **1**

(iv) Deduce the value of k required for $ABMN$ to be a parallelogram. **2**

(b) (i) Express $\dfrac{x^2+1}{(x-1)\left(x^2+x+1\right)} \equiv \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$. **2**

(ii) Hence find $\displaystyle\int \frac{x^2+1}{(x-1)\left(x^2+x+1\right)}\,dx$ **3**

(c) Prove using contradiction that $\log_2 5$ is an irrational number. **3**

End of Question 12

Question 13 (15 marks) Start a new writing booklet.

(a) Consider a sphere S, with centre $C(2,-1,0)$ and radius $\sqrt{29}$ units.
Consider also the line l with parametric equations

$$x=\lambda+1,\ y=\lambda,\ z=2\lambda+3$$

(i) Find the vector equation of line l, writing your answer in the form $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{b}$, where $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are expressed as column vectors. **1**

It is also known that l intersects the surface of S at points P and Q

(ii) Find the coordinates of points P and Q. **3**

(iii) Hence or otherwise, determine whether PQ is a diameter of S, showing necessary working. **1**

(b) The displacement, x metres, of a particle P from the origin O at time t seconds is given by

$$x=6\cos(2t+\frac{\pi}{4})+\cos(2t)$$

(i) Show that P is moving in simple harmonic motion about O **3**

(ii) Find the amplitude of this motion, correct to 1 decimal place. **3**

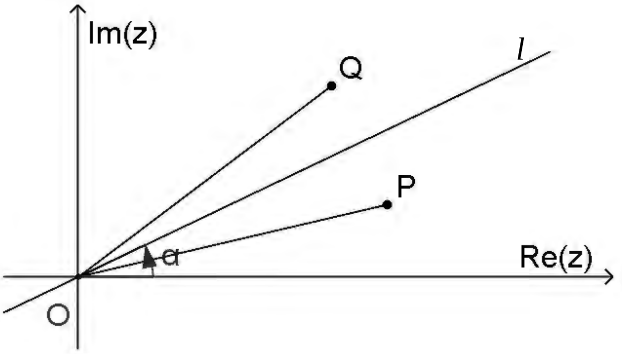
(c) Using Trigonometric substitution, or otherwise, find $\int\frac{1}{\sqrt{(1-x^2)^3}}dx$. **4**

Give your answer without trigonometric functions.

End of Question 13

QUESTION 14 (15 marks)

(a) Let l be the line in the complex plane that passes through the origin and makes an angle α with the positive real axis, where $0 < \alpha < \frac{\pi}{2}$.



The point P represents the complex number z_1, where $0 < \arg(z_1) < \alpha$. The point P is reflected in the line l to produce the point Q, which represents the complex number z_2. Hence $|z_1| = |z_2|$.

(i) Explain why $\arg(z_1) + \arg(z_2) = 2\alpha$. **2**

(ii) Deduce that $z_1 z_2 = |z_1|^2 (\cos 2\alpha + i \sin 2\alpha)$. **1**

(iii) Let $\alpha = \frac{\pi}{4}$ and let R be the point that represents the complex number $z_1 z_2$. **1**
Describe the locus of R as z_1 varies.

(b) If a, b and c are real and unequal and that $a^2 + b^2 > 2ab$ deduce that

(i) $a^2 + b^2 + c^2 > ab + bc + ac.$ **1**

(ii) If $a + b + c = 6$ show that $ab + bc + ac < 12.$ **2**

Question 14 continues over the page

(c) Use the substitution of $t = \tan\frac{x}{2}$, find $\int \frac{1}{3-\cos x - 2\sin x}\, dx$ **3**

(d) A particle, initially at $x = 2$, has a velocity given by $v = \sqrt{16-3x^2}$. Find the expression of x in terms of t. **3**

(e) If the displacement of an object at any time t is given by the vector equation **2**

$$\underset{\sim}{r} = \begin{bmatrix} 10\cos t \\ 10\sin t \\ 15-t \end{bmatrix}$$

Find the vector equation of the velocity and its initial speed.

End of Question 14

Question 15 (15 marks) Start a new writing booklet.

(a) Consider the equation $z^5 + 1 = 0$, where z is a complex number.

(i) Solve the equation $z^5 + 1 = 0$ by finding the 5th roots of -1. **2**

(ii) Show that if z is a solution of $z^5 + 1 = 0$ and $z \neq 1$, then $u = z + \frac{1}{z}$ is a solution of $u^2 - u - 1 = 0$. **2**

(iii) Hence find the exact value of $\cos\frac{3\pi}{5}$. **3**

(b) Consider the polynomial equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

Where a, b, c and d are all integers.
Suppose also that the equation has a root of the form ki, where k is real, and $k \neq 0$.

(i) State why the conjugate, $-ki$ is also a root. **1**

(ii) Show that $c = k^2 a$. **2**

(iii) Show that $c^2 + a^2 d = abc$. **2**

(iv) If 2 is also a root of the equation, and $b = 0$, show that c is even. **3**

End of Question 15

QUESTION 16 (15 marks)

(a) Consider the recurrence relation defined by $T_1 = 1$ and

$$T_{n+1} = \frac{4+T_n}{1+T_n}, \quad \text{for } n = 1, 2, 3, \ldots$$

(i) Prove by mathematical induction that for $n \geq 1$, **4**

$$T_n = 2\left[\frac{1+(-3)^{-n}}{1-(-3)^{-n}}\right].$$

(ii) Hence find the limit value of T_n as $n \to \infty$. **1**

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \; dx$ and let $J_n = (-1)^n I_{2n}$

(i) Show that $I_n + I_{n+2} = \frac{1}{n+1}$. **2**

(ii) Find the value of $I_4 = \int_0^{\frac{\pi}{4}} \tan^4 x \; dx$ **2**

(iii) Deduce that $J_n - J_{n-1} = \frac{(-1)^n}{2n-1}$ for $n \geq 1$. **1**

(iii) By considering **2**

$$J_m = (J_m - J_{m-1}) + (J_{m-1} - J_{m-2}) + (J_{m-2} - J_{m-3}) + \ldots + (J_1 - J_0) + J_0$$

Show that

$$J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}.\text{for } n \geq 1.$$

(v) Use substitution $u = \tan x$ to show that $I_n = \int_0^1 \frac{u^n}{1+u^2} \; du$. **1**

(vi) Deduce that $0 \leq I_n \leq \frac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$. **2**

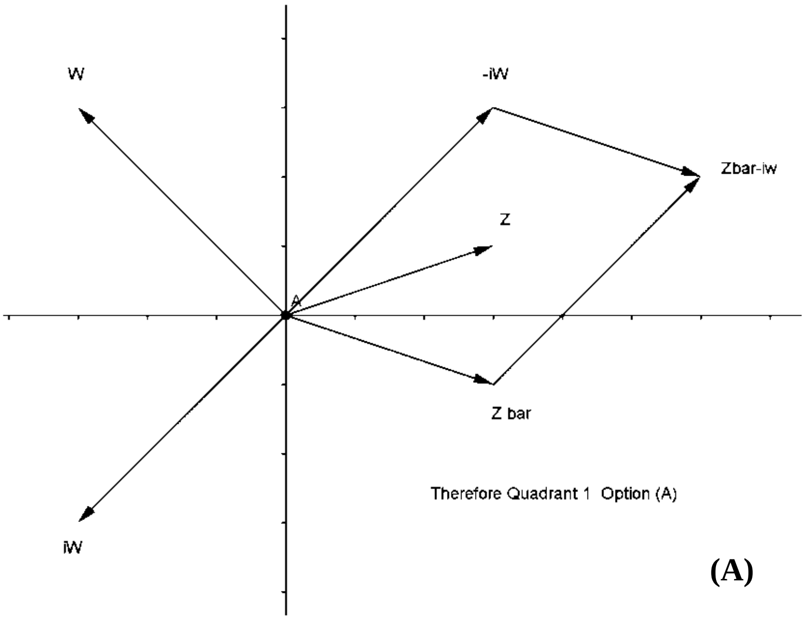
End of Examination

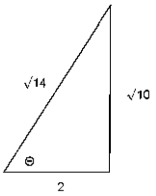
2024 TRIAL EXAMINATION FOR EXTENSION 2

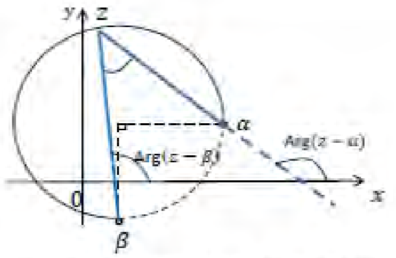
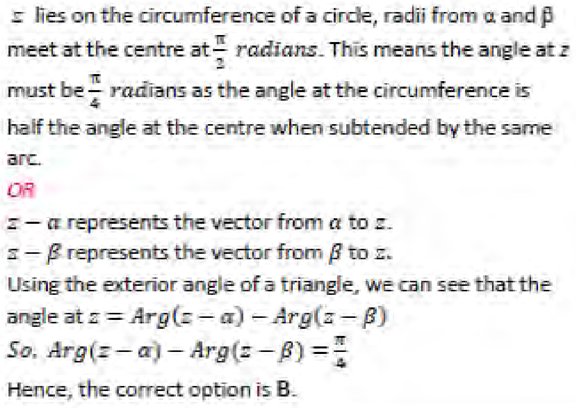
Section I Multiple Choice

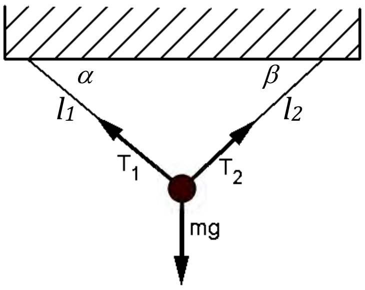
Question	Answers
1	**C**
2	**B**
3	**C**
4	**D**
5	**A**
6	**B**
7	**D**
8	**D**
9	**B**
10	**C**

1.	$\int \frac{-1}{\sqrt{1-16x^2}}dx = \frac{1}{4}\int \frac{-1}{\sqrt{\frac{1}{16}-x^2}}dx$ $= \frac{1}{4}\cos^{-1}\frac{x}{\left(\frac{1}{4}\right)}+C$ $= \frac{1}{4}\cos^{-1}4x+C$ **(C)**	90% correct.
2.	$(\underset{\sim}{b}-\underset{\sim}{a})=\lambda(\underset{\sim}{c}-\underset{\sim}{a})$ has more information than $\overrightarrow{AB} \parallel \overrightarrow{AC}$.  Point B is diving the interval AC in the ratio of $\lambda:(1-\lambda)$ internally. **(B)**	61% correct. Popular choice was A. Question required false statement. Upon reflection we will allow **(c)** as well as a correct answer
3.	(a) $\forall$ *means" for all"* $a=30^{\circ}$ $b=150^{\circ}$ *not true* (b) $\forall$ *means" for all"* $a=5$ $b=-3$ *not true* (c) $\exists$ *means"there exists"* $a=2$ $b=2$ *is true* (d) $\exists$ *means"there exists"* $\lvert a+b\rvert > \lvert a\rvert+\lvert b\rvert$ *not true as* $\lvert a\rvert+\lvert b\rvert \geq \lvert a+b\rvert$ *by* Δ *inequality* $\forall\mathbb{C}$ **(C)**	74% correct. Popular choice was D. Third side of a triangle is less than the sum of the two sides.

4.	$\int x\tan^{-1}x\,dx = uv - \int u'v\,dx$ where $u=\tan^{-1}x$ $v'=x$ $u'=\frac{1}{1+x^2} \longleftarrow v=\frac{x^2}{2}$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2+1-1}{1+x^2}\,dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int\frac{x^2+1}{1+x^2}-\frac{1}{1+x^2}\,dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int 1-\frac{1}{1+x^2}\,dx$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\left(x-\tan^{-1}x\right)+C$ $=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+C$ **(D)**	100% correct.
5.	 **(A)**	87% correct. Popular choice was D.

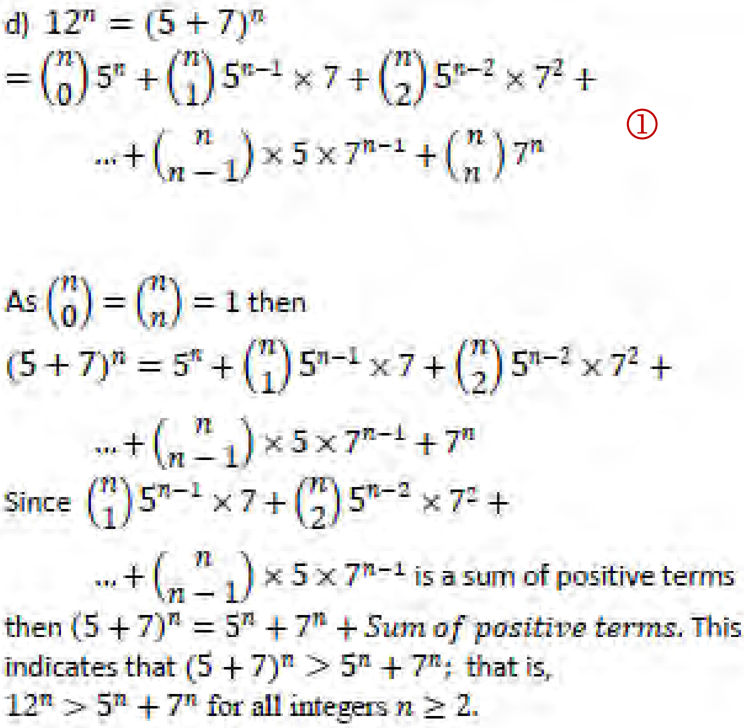
6.	Original statement: 'For any function $f\left(x\right)$, $f\left(x\right)$ is not continuous at $x=a\Rightarrow$ $f\left(x\right)$ is not differentiable at $x=a$.' Converse: For any function $f\left(x\right)$, $f\left(x\right)$ is not differentiable at $x=a\Rightarrow$ $f\left(x\right)$ is not continuous at $x=a$.' False. Contrapositive: For any function $f\left(x\right)$, $f\left(x\right)$ is differentiable at $x=a\Rightarrow$ $f\left(x\right)$ is continuous at $x=a$.' True. **(B)**	68% correct. Popular choice was D.
7.	$\left\lvert\overrightarrow{OA}\right\rvert=\sqrt{4+1+9}=\sqrt{14}$ As $\overrightarrow{OA}.\overrightarrow{OB}=\left\lvert\overrightarrow{OA}\right\rvert.\left\lvert\overrightarrow{OB}\right\rvert\cos\theta$ then $6=\sqrt{14}\times3\times\cos\theta$ $\therefore\cos\theta=\dfrac{2}{\sqrt{14}}$ Since $\overrightarrow{OA}.\overrightarrow{OB}>0$ and θ is in a triangle, then θ is acute. Using the triangle, $\sin\theta=\dfrac{\sqrt{10}}{\sqrt{14}}$  $\therefore Area=\dfrac{1}{2}\times3\times\sqrt{14}\times\dfrac{\sqrt{10}}{\sqrt{14}}$ $=\dfrac{3\sqrt{10}}{2}$ (D)	81% correct. Popular choice was B.

8.	$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\ dx$ $=\int_0^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\ dx$ $=\int_0^{\pi}\frac{\pi\sin(\pi-x)}{1+\cos^2(\pi-x)}-\frac{x\sin(\pi-x)}{1+\cos^2(\pi-x)}\ dx$ $=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}-\frac{x\sin x}{1+\cos^2 x}\ dx$ $=\int_0^{\pi}\frac{\pi\sin x}{1+\cos^2 x}\ dx-\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\ dx$ $I=\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\ dx-I$ $2I=\pi\left(\frac{\pi}{2}\right)$ $2I=\frac{\pi^2}{2}$ $\therefore I=\frac{\pi^2}{4}$ **(D)**	Poorly done. 48% correct. Popular choice was C. Students forgotten it was $2I$ that equals $\frac{\pi^2}{2}$.
9.	  z lies on the circumference of a circle, radii from α and β meet at the centre at $\frac{\pi}{2}$ $radians$. This means the angle at z must be $\frac{\pi}{4}$ $radians$ as the angle at the circumference is half the angle at the centre when subtended by the same arc. *OR* $z-\alpha$ represents the vector from α to z. $z-\beta$ represents the vector from β to z. Using the exterior angle of a triangle, we can see that the angle at $z=Arg(z-\alpha)-Arg(z-\beta)$ *So.* $Arg(z-\alpha)-Arg(z-\beta)=\frac{\pi}{4}$ Hence, the correct option is B. **(B)**	100% correct.

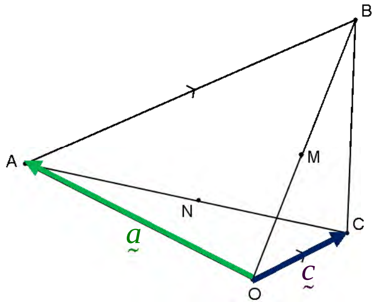
10.	 When $l_1 > l_2$, then $\beta > \alpha$ and the shorter string will experience greater tension, i.e. $T_2 > T_1$. **(C)**	Poorly done. 58% correct. Popular choice was D.

SECTION 2

11.	(a) $w+\bar{z}=8-2i+(-5-3i)$ ① $=3-5i$	Well done
11.	(b)(i) $(1-2i)^2=1-4i-4$ ① $=-3-4i$	Well done
11.	(b)(ii) ① $z=\dfrac{5\pm\sqrt{(-5)^2-4.1.(7+i)}}{2.1}$ $=\dfrac{5\pm\sqrt{25-4(7+i)}}{2}$ $=\dfrac{5\pm\sqrt{-3+4i}}{2}$ ① $=\dfrac{5\pm\sqrt{(1-2i)^2}}{2}, \textit{from}(i)$ $=\dfrac{5\pm(1-2i)}{2}$ $=3-i,2+i$ ①	Well done
11.	(c)(i) $\dfrac{d}{dx}(x\sin^{-1}x)=x.\dfrac{1}{\sqrt{1-x^2}}+\sin^{-1}x.1$ $=\dfrac{x}{\sqrt{1-x^2}}+\sin^{-1}x$ ①	Well done
11.	(c)(ii) $\int\dfrac{d}{dx}(x\sin^{-1}x)dx=\int\dfrac{x}{\sqrt{1-x^2}}dx+\int\sin^{-1}(x)dx$ $\therefore\int\sin^{-1}(x)dx=\int\dfrac{d}{dx}(x\sin^{-1}x)dx-\int\dfrac{x}{\sqrt{1-x^2}}dx$ ① $=x\sin^{-1}x+\sqrt{1-x^2}+C$ ①	Mostly well done

11.	(d)  d) $12^n = (5+7)^n$ $= \binom{n}{0}5^n + \binom{n}{1}5^{n-1} \times 7 + \binom{n}{2}5^{n-2} \times 7^2 +$ $\dots + \binom{n}{n-1} \times 5 \times 7^{n-1} + \binom{n}{n}7^n$ ① As $\binom{n}{0} = \binom{n}{n} = 1$ then $(5+7)^n = 5^n + \binom{n}{1}5^{n-1} \times 7 + \binom{n}{2}5^{n-2} \times 7^2 +$ $\dots + \binom{n}{n-1} \times 5 \times 7^{n-1} + 7^n$ Since $\binom{n}{1}5^{n-1} \times 7 + \binom{n}{2}5^{n-2} \times 7^2 +$ $\dots + \binom{n}{n-1} \times 5 \times 7^{n-1}$ is a sum of positive terms ① then $(5+7)^n = 5^n + 7^n + \mathit{Sum\ of\ positive\ terms}$. This indicates that $(5+7)^n > 5^n + 7^n$; that is, $12^n > 5^n + 7^n$ for all integers $n \geq 2$.	Mostly well done
11.	(e)(i) $e^{in\theta} + e^{-in\theta} = [\cos(n\theta) + i\sin(n\theta)] + [\cos(-n\theta) + i\sin(-n\theta)]$ $= \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$ ① $= 2\cos(n\theta)$	well done
11.	(e)(ii) $(e^{i\theta} + e^{-i\theta})^4 = e^{i4\theta} + 4e^{i2\theta} + 6 + 4 + 4e^{-i2\theta} + e^{-i4\theta}$ $= (e^{i4\theta} + e^{-i4\theta}) + 4(e^{i2\theta} + e^{-i2\theta}) + 6$ $\therefore (2\cos\theta)^4 = 2\cos 4\theta + 4(2\cos 2\theta) + 6$ ① $\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$ $\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ①	well done

11.	(e)(iii) $\int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta = \int_0^{\frac{\pi}{2}} (\frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8})d\theta$ $= \left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3\theta}{8}\right]_0^{\frac{\pi}{2}}$ $= \left[\frac{1}{32}\sin 4\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{2}\right) + \frac{3\left(\frac{\pi}{2}\right)}{8}\right] - \left[\frac{1}{32}\sin 4(0) + \frac{1}{4}\sin 2(0) + \frac{3(0)}{8}\right]$ $= \frac{3\pi}{16}$	Mostly well done ① ①

12.	(a)(i)  $\overrightarrow{OM} = \frac{1}{2}\overrightarrow{OB}$ $= \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{AB}\right)$ but $\overrightarrow{AB} = k\overrightarrow{OC}$ ① for $\overrightarrow{OB}$ $= \frac{1}{2}\left(\overrightarrow{OA} + k\overrightarrow{OC}\right)$ $\therefore \overrightarrow{OM} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}k\underset{\sim}{c}$ ① for $\overrightarrow{OM}$	Mostly did well!
12.	(a)(ii) $\overrightarrow{ON} = \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CA}$ $= \overrightarrow{OC} + \frac{1}{2}\left(\overrightarrow{OA} - \overrightarrow{OC}\right)$ ① for $\overrightarrow{CA}$ $= \underset{\sim}{c} + \left(\frac{1}{2}\underset{\sim}{a} - \frac{1}{2}\underset{\sim}{c}\right)$ $\therefore \overrightarrow{ON} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$ ① for $\overrightarrow{ON}$ OR $\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right)$ ① for $\overrightarrow{CA}$ $\therefore \overrightarrow{ON} = \frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}$ ① for $\overrightarrow{ON}$	Mostly did well!
12.	(a)(iii) $\therefore \overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$ $= \left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}\underset{\sim}{c}\right) - \left(\frac{1}{2}\underset{\sim}{a} + \frac{1}{2}k\underset{\sim}{c}\right)$	All did well!

	$= \frac{1}{2}\underset{\sim}{c} - \frac{1}{2}k\underset{\sim}{c}$ $\therefore \overrightarrow{MN} = \frac{1}{2}\underset{\sim}{c}(1-k)$ ① for $\overrightarrow{MN}$	
12.	(a)(iv) For *ABMN* to be a parallelogram, $\overrightarrow{MN} = \overrightarrow{BA}$ ① $\frac{1}{2}\underset{\sim}{c}(1-k) = -k\underset{\sim}{c}$ $\frac{1}{2}(1-k) = -k$ $1-k = -2k$ $1-k = 2k$ $1 = -k$ $k = -1$ ① for k OR For *ABMN* to be a parallelogram, such that $\lvert\overrightarrow{MN}\rvert = \lvert-\overrightarrow{AB}\rvert$ ① $\left\lvert\frac{1}{2}\underset{\sim}{c}(1-k)\right\rvert = k\lvert-\underset{\sim}{c}\rvert$ $\frac{1}{2}\lvert(1-k)\rvert\lvert\underset{\sim}{c}\rvert = k\lvert-\underset{\sim}{c}\rvert$ since $1-k<0,\ k>1$ $\therefore \lvert(1-k)\rvert = k-1$ $\frac{1}{2}(k-1) = k$ $k-1 = 2k$ $-1 = k$ $\therefore k = -1$ ① for k	Not as well executed. Students who paid attention to direction of vectors get the correct answers.
12.	(b)(i) $\frac{x^2+1}{(x-1)(x^2+x+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ $x^2+1 \equiv A(x^2+x+1) + (Bx+C)(x-1)$ When $x=1$, $(1)^2+1 \equiv A((1)^2+(1)+1)+0$ $2 = 3A$ $A = \frac{2}{3}$ When $x=0$, $(0)^2+1 \equiv \frac{2}{3}((0)^2+(0)+1)+(B(0)+C)((0)-1)$ $1 = \frac{2}{3} - C$ $C = \frac{2}{3} - 1$ $C = -\frac{1}{3}$	Mostly did well! A few arithmetic calculation error. A few students did not answer as required in $\frac{x^2+1}{(x-1)(x^2+x+1)} \equiv \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$ form.

When $x=2,\ (2)^2+1\equiv\frac{2}{3}\left((2)^2+(2)+1\right)+\left(B(2)-\frac{1}{3}\right)\left((2)-1\right)$

$$5=\frac{14}{3}+2B-\frac{1}{3}$$

$$5=\frac{13}{3}+2B$$

$$2B=\frac{2}{3}$$

$$B=\frac{1}{3}$$

$$\therefore\frac{x^2+1}{(x-1)\left(x^2+x+1\right)}\equiv\frac{\frac{2}{3}}{x-1}+\frac{\frac{1}{3}x-\frac{1}{3}}{x^2+x+1}$$ ① ①

12. (b)(ii) $\displaystyle\int\frac{x^2+1}{(x-1)\left(x^2+x+1\right)}\,dx$

$$=\int\frac{\frac{2}{3}}{x-1}+\frac{\frac{1}{3}x-\frac{1}{3}}{x^2+x+1}\,dx$$

$$=\int\frac{\frac{2}{3}}{x-1}\,dx+\int\frac{\frac{1}{3}x-\frac{1}{3}}{x^2+x+1}\,dx$$

$$=\frac{2}{3}\int\frac{1}{x-1}\,dx+\frac{1}{3}\int\frac{x-1}{x^2+x+1}\,dx$$

①

$$=\frac{2}{3}\ln|x-1|+\frac{1}{2}\left(\frac{1}{3}\right)\int\frac{2x-2}{x^2+x+1}\,dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\int\frac{2x+1-2-1}{x^2+x+1}\,dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\int\frac{2x+1}{x^2+x+1}-\frac{3}{x^2+x+1}\,dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\int\frac{2x+1}{x^2+x+1}\,dx-\frac{3}{6}\int\frac{1}{x^2+x+1}\,dx$$

①

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\int\frac{1}{x^2+x+1}\,dx$$

$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\int\frac{1}{x^2+x+\left(\frac{1}{2}\right)^2+1-\frac{1}{4}}\,dx$$

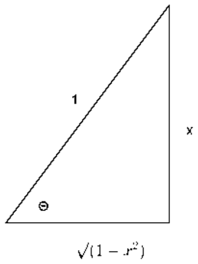
$$=\frac{2}{3}\ln|x-1|+\frac{1}{6}\ln\left(x^2+x+1\right)-\frac{1}{2}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$$

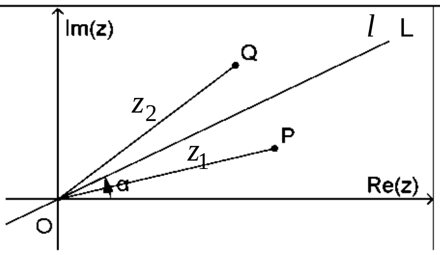
	$= \frac{2}{3}\ln\lvert x-1\rvert + \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)\int \frac{1\times\frac{\sqrt{3}}{2}}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$ $= \frac{2}{3}\ln\lvert x-1\rvert + \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}+C$ $= \frac{2}{3}\ln\lvert x-1\rvert + \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)+C$ ① $= \frac{2}{3}\ln\lvert x-1\rvert + \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+C$	
12.	(c) Assume that there exists $p, q \in N$ such that $\log_2 5 = \frac{p}{q}$ and the highest common factor of p and q is 1. ① $5 = 2^{\frac{p}{q}}$ $(5)^q = \left(2^{\frac{p}{q}}\right)^q$ $\therefore 5^q = 2^p$ But LHS $= 5^q$ $= 5\times5\times5\times\ldots$ i.e. 5 is a factor of 5^q but not 2^p and RHS $= 2^p$ $= 2\times2\times2\times\ldots$ i.e. 2 is a factor of 2^p but not 5^q ① Then no p and q such that $p, q \in N$ satisfies the equation $5^q = 2^p$ and this equation must be a contradiction. So $\log_2 5$ is an irrational number. ①	

13.	(a)(i) $\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda+1 \\ \lambda \\ 2\lambda+3 \end{pmatrix}$ ① $= \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	well done

13.	(a)(ii) Equation of sphere $\left\|\underset{\sim}{r}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right\|=\sqrt{29}$ $\left\|\begin{pmatrix}1\\0\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right\|=\sqrt{29}$ $\left\|\lambda\begin{pmatrix}1\\1\\2\end{pmatrix}+\begin{pmatrix}-1\\1\\3\end{pmatrix}\right\|=\sqrt{29}$ $\therefore(\lambda-1)^2+(\lambda+1)^2+(2\lambda+3)^2=29$ $\therefore 6\lambda^2+12\lambda-18=0$ $\therefore \lambda^2+2\lambda-3=0$ $\therefore(\lambda+3)(\lambda-1)=0$ $\therefore \lambda=-3,1$ $\lambda=-3: P=(1-3,0-3,3-6)=(-2,-3,-3)$ $\lambda=1: Q=(1+1,0+1,3+2)=(2,1,5)$	Mostly well done ① ① ①
13.	(a)(iii) $PQ=\sqrt{(-2-2)^2+(-3-1)^2+(-3-5)^2}$ $=\sqrt{16+16+64}$ $=\sqrt{96}$ ① $=2\sqrt{24}$ $\neq 2\sqrt{29}$ Where $\sqrt{29}$ is the radius $\therefore PQ$ is not the diameter of the sphere.	Well done

13.	(b)(i) $x = 6\cos(2t+\frac{\pi}{4})+\cos(2t)$ $\dot{x} = -12\sin(2t+\frac{\pi}{4})-2\sin(2t)$ $\ddot{x} = -24\cos(2t+\frac{\pi}{4})-4\cos(2t)$ $= -4\left[6\cos(2t+\frac{\pi}{4})+\cos(2t)\right]$ $\ddot{x} = -4x$ $\ddot{x} = -n^2x$ Which is SHM about $c=0$ and $n=2$ $\therefore$ period = $\frac{2\pi}{2}=\pi$	Mostly well done ① ① ①
13.	(b)(ii) $x = 6\cos(2t+\frac{\pi}{4})+\cos(2t)$ $= 6\left[\cos(2t)\cos\frac{\pi}{4}-\sin(2t)\sin\frac{\pi}{4}\right]+\cos(2t)$ $= \frac{6\sqrt{2}}{2}[\cos(2t)-\sin(2t)]+\cos(2t)$ $= (3\sqrt{2}+1)\cos(2t)-3\sqrt{2}\sin(2t)$ $= R\cos(2t+\alpha)$ *where* $R=\sqrt{(3\sqrt{2}+1)^2+(3\sqrt{2})^2}$ $= \sqrt{6\sqrt{2}+37}$ $= 6.744$ $\therefore amplitude = 6.7\ (1d.p)$	Quite a few students did not realise that the amplitude could be found by simply ①finding the coefficient of the distance equation by using the transformation/auxiliary angle formula. This was a lot easier than finding when the velocity ①was zero and substituting back t into x. ①

13.	(c) $x = \sin\theta$ $dx = \cos\theta$ $\therefore \int \frac{dx}{(1-x^2)^3} = \int \frac{\cos\theta . d\theta}{\sqrt{(1-\sin^2\theta)^3}}$ let $= \int \frac{\cos\theta . d\theta}{(\cos^2\theta)^{\frac{3}{2}}}$ $= \int \frac{d\theta}{\cos^2\theta}$ $= \int \sec^2\theta . d\theta$ $= \tan\theta + C$ $= \frac{x}{\sqrt{1-x^2}}+C$ 	Very well done ① ① ① ①

14.	(a)(i)  Let the line l passes through the point L such that $\angle LOP = \angle LOQ = \alpha - \arg(z_1)$ ① for any similar statement $\arg(z_2) = \angle QOL + \alpha$ $= (\alpha - \arg(z_1)) + \alpha$ $= 2\alpha - \arg(z_1)$ $\therefore \arg(z_1) + \arg(z_2) = 2\alpha$ (as required) ① including alternate methods	Mostly did well!
14.	(a)(ii) $\lvert z_1 z_2\rvert = \lvert z_1\rvert \lvert z_2\rvert$ and since $\lvert z_1\rvert = \lvert z_2\rvert$ $\therefore \lvert z_1 z_2\rvert = \lvert z_1\rvert^2$ $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg(z_1 z_2) = 2\alpha$ [from part (i)] $\therefore z_1 z_2 = \lvert z_1\rvert^2 (\cos 2\alpha + i \sin 2\alpha)$. (as required) ①	
14.	(a)(iii) When $\alpha = \frac{\pi}{4}$, $z_1 z_2 = \lvert z_1\rvert^2 \left(\cos 2\left(\frac{\pi}{4}\right) + i \sin 2\left(\frac{\pi}{4}\right)\right)$ ① $= \lvert z_1\rvert^2 \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ $z_1 z_2 = \lvert z_1\rvert^2 i$. $\lvert z_1\rvert \neq 0$ otherwise $\arg(z_1)$ is undefined. (So the locus of R is dependent on $\lvert z_1\rvert$). Hence the locus of R is the positive y – axis, excluding the origin, of length $\lvert z_1\rvert^2$ ①	Many have problem with the locus of R. Few students answered partially correct not taking into account that $\lvert z_1\rvert^2$ will only give positive values and forget that O should be excluded.
14.	(b)(i) $a^2 + b^2 > 2ab$ — ① $b^2 + c^2 > 2bc$ — ② $a^2 + c^2 > 2ac$ — ③ ① + ② + ③ $2a^2 + 2b^2 + 2c^2 > 2ab + 2bc + 2ac$ $2(a^2 + b^2 + c^2) > 2(ab + bc + ac)$ $\therefore a^2 + b^2 + c^2 > ab + bc + ac$ (as required) ①	All did well.

14.	(b)(ii) Since $a+b+c=6$ $(a+b+c)^2=(6)^2$ $(a+b+c)^2=36$ $[(a+b)+c]^2=36$ $(a+b)^2+2(a+b)c+c^2=36$ $a^2+2ab+b^2+2ac+2bc+c^2=36$ $(a^2+b^2+c^2)+2ab+2ac+2bc=36$ — ④ ① And from (i) $a^2+b^2+c^2>ab+bc+ac$ $a^2+b^2+c^2+2(ab+bc+ac)>ab+bc+ac+2(ab+bc+ac)$ $a^2+b^2+c^2+2(ab+bc+ac)>3(ab+bc+ac)$ — ⑤ Sub ④ into ⑤ $36>3(ab+bc+ac)$ $\therefore ab+bc+ac<12$ ①	Mostly did well.
14.	(c) $\int \frac{1}{3-\cos x-2\sin x}\,dx$ Let $t=\tan\frac{x}{2}$ $\frac{x}{2}=\tan^{-1}t$ $x=2\tan^{-1}t$ $\frac{dx}{dt}=\frac{2}{1+t^2}$ $dx=\frac{2}{1+t^2}\,dt$ And $\cos x=\frac{1-t^2}{1+t^2}$ $\sin x=\frac{2t}{1+t^2}$ $=\int \frac{1}{3-\frac{1-t^2}{1+t^2}-2\left(\frac{2t}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt$ ① $=\int \frac{1}{\frac{3(1+t^2)-(1-t^2)-2(2t)}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt$ $=\int \frac{1+t^2}{3(1+t^2)-(1-t^2)-2(2t)}\left(\frac{2}{1+t^2}\right)dt$ $=\int \frac{1+t^2}{3+3t^2-1+t^2-4t}\left(\frac{2}{1+t^2}\right)dt$ $=\int \frac{2}{4t^2-4t+2}\,dt$ ①	Mostly did well. Some errors were: 1) answer in term of t and not x. 2) forgotten the existence of the first term 3 when working out the equivalent fraction with denominator $1+t^2$. 3) was stuck with working out $\int \frac{2}{4t^2-4t+2}\,dt$

	$= \int \frac{2}{\left(4t^2-4t+1\right)+2-1}\ dt$ $= \int \frac{2}{(2t-1)^2+1}\ dt$ i.e. $f(x)=2x-1$ and $a=1$ $f'(x)=2$ $= \tan^{-1}(2t-1)+C$ $= \tan^{-1}\left(2\tan\frac{x}{2}-1\right)+C$ ①	
14.	(d) $v=\sqrt{16-3x^2}$ $\frac{dx}{dt}=\sqrt{16-3x^2}$ $\frac{1}{\sqrt{16-3x^2}}\,dx=dt$ $\int_2^x \frac{1}{\sqrt{16-3x^2}}\,dx=\int_0^t dt$ ① $\int_2^x \frac{1}{\sqrt{(4)^2-\left(\sqrt{3}x\right)^2}}\,dx=[t]_0^t$ $\frac{1}{\sqrt{3}}\int_2^x \frac{\sqrt{3}}{\sqrt{(4)^2-\left(\sqrt{3}x\right)^2}}\,dx=t-0$ $\frac{1}{\sqrt{3}}\left[\sin^{-1}\frac{\sqrt{3}x}{4}\right]_2^x=t$ $\sin^{-1}\frac{\sqrt{3}x}{4}-\sin^{-1}\frac{\sqrt{3}(2)}{4}=\sqrt{3}t$ $\sin^{-1}\frac{\sqrt{3}x}{4}-\sin^{-1}\frac{\sqrt{3}}{2}=\sqrt{3}t$ $\sin^{-1}\frac{\sqrt{3}x}{4}-\frac{\pi}{3}=\sqrt{3}t$ ① $\sin^{-1}\frac{\sqrt{3}x}{4}=\sqrt{3}t+\frac{\pi}{3}$ $\frac{\sqrt{3}x}{4}=\sin\left(\sqrt{3}t+\frac{\pi}{3}\right)$ $\therefore x=\frac{4}{\sqrt{3}}\sin\left(\sqrt{3}t+\frac{\pi}{3}\right)$ ①	Mostly did well especially using the boundaries.
14.	(e) $\underset{\sim}{r}(t)=\begin{bmatrix}10\cos t\\10\sin t\\15-t\end{bmatrix}$	Mostly did well and a handful of students found the initial velocity vector but forgot to calculate the speed.

	$\therefore \underset{\sim}{v}(t)=\begin{bmatrix}-10\sin t\\10\cos t\\-1\end{bmatrix}$ ① $\underset{\sim}{v}(0)=\begin{bmatrix}-10\sin(0)\\10\cos(0)\\-1\end{bmatrix}$ $\underset{\sim}{v}(0)=\begin{bmatrix}0\\10\\-1\end{bmatrix}$ $\lvert\underset{\sim}{v}(0)\rvert=\sqrt{0^2+(10)^2+(-1)^2}$ $\lvert\underset{\sim}{v}(0)\rvert=\sqrt{101}\,\text{ms}^{-1}$ ①	

15.	(a)(i) $z^5+1=0$ $\therefore z^5=-1\Rightarrow e^{i(\pi+2\pi n)}$ *where* $n\in\mathbb{Z}$ *So* $z=e^{i\left[\frac{(\pi+2\pi n)}{5}\right]}$ *where* $n=0,1,-1,2,-2.$ $\Rightarrow z=e^{\frac{i\pi}{5}},e^{-\frac{i\pi}{5}},e^{\frac{3i\pi}{5}},e^{-\frac{3i\pi}{5}},-1$	Mostly well done ①①
15.	(a)(ii) $z^5+1=0$ $(z+1)(z^4-z^3+z^2-z+1)=0$ *now* $z\neq-1$ $\therefore z^4-z^3+z^2-z+1=0$ $\therefore \frac{z^4-z^3+z^2-z+1}{z^2}=0$ $\therefore z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$ $\therefore z^2+2+\frac{1}{z^2}-(z+\frac{1}{z})-1=0$ $\therefore (z+\frac{1}{z})^2-(z+\frac{1}{z})-1=0$ *sub* $u=z+\frac{1}{z}$ $\therefore u^2-u-1=0$ *as req.*	Mostly well done ① ①

15.	(a)(iii) $u^2-u-1=0$ $\therefore u=\frac{1\pm\sqrt{5}}{2}$ $now\ u=z+\frac{1}{z}$ $=z+\bar{z}$ $=2\,\mathrm{Re}(z)$ $=2\cos\theta$ $Hence\ \cos\theta=\frac{u}{2}=\frac{1\pm\sqrt{5}}{4}$ $Now\ \cos\frac{\pi}{5}>\cos\frac{3\pi}{5}$ $(1st\ quad\ pos>2nd\ quad\ neg)$ $\therefore\cos\frac{3\pi}{5}=\frac{1-\sqrt{5}}{4}$	Mostly well done Some students missed the correct quadrant and did ①the + instead of the – ① ①
15.	(b)(i) If a complex number $a+ib$ where a,b are real, is a root of a polynomial equation with real coefficienta, then its complex conjugate is also a root of this equation. Since a,b,c,d are integer coefficients as stated, then the conjugate of $0+ki$, being $0-ki$ is also a root.	well done ①
15.	(b)(ii) Since $x=ki$ is a root of $x^4+ax^3+bx^2+cx+d=0$ Then $(ki)^4+a(ki)^3+b(ki)^2+c(ki)+d=0$ i.e. $k^4-ak^3i-bk^2+cki+d=0$ i.e. $(k^4-bk^2+d)+i(ck-ak^3)=0+0i$ $\therefore ck-ak^3=0\Rightarrow k(c-ak^2)=0\ \ equating\ imag.\ parts$ $\therefore c=ak^2\ \ (as\,k\neq 0)$ as required.	Mostly well done. Some used a different method ①which was fine but led to problems as not knowing how to proceed correctly in parts (iii) and (iv) ①
	(b)(iii) From (ii) $k^4-bk^2+d=0$ (equating real) and since $c=ak^2\Rightarrow k^2=\frac{c}{a}$ Then $(\frac{c}{a})^2-b(\frac{c}{a})+d=0$ $c^2-abc+a^2d=0$ $\therefore c^2+a^2d=abc$ as req.	Mostly well done if used this method’ ① ①
	(b)(iv) If $b=0$, then from (iii) $c^2+a^2d=0$ then $d=-(\frac{c}{a})^2$ and since d is integral, then c must be divisible by a, i.e $c=ap$ where p is integral.	Mostly well done if used this method. There is no guarantee that roots are integral in saying

	Then $d = -(\frac{ap}{a})^2 = -p^2$ Also, the given equation $x^4 + ax^3 + bx^2 + cx + d = 0$ becomes $16 + 8a + 2c + d = 0$ since 2 is a root. From $16 + 8a + 2c + d = 0 \Rightarrow d = -2(8 + 4a + c)$ hence d is even as it divisible by 2 and $8, 4, a, c$ are integral. Since d is even and $d = -p^2$, it follows p must be even and hence $p = 2q \ \therefore d = -4q^2$ Now from $16 + 8a + 2c + d = 0$ we have $16 + 8a + 2c - 4q^2 = 0$ That is $c = -2(q^2 - 4 - 2a)$ which is even as required as q, a are integral.	①it is divisible by 2 which quite a few stated. i.e α and β etc ① ①

16.	(a)(i) $T_n = 2\left[\frac{1+(-3)^{-n}}{1-(-3)^{-n}}\right]$ When $n = 1$, $T_1 = 2\left[\frac{1+(-3)^{-1}}{1-(-3)^{-1}}\right]$ $= 2\left[\frac{1-\frac{1}{3}}{1-\left(-\frac{1}{3}\right)}\right]$ $= 2\left[\frac{\left(\frac{2}{3}\right)}{\left(\frac{4}{3}\right)}\right]$ $= 2\left[\frac{1}{2}\right]$ $\therefore T_1 = 1$ which agrees with the recursive definition. Hence true for $n = 1$. ① Assume true for $n = 1$ to $n = k$, $T_1 = 2\left[\frac{1+(-3)^{-1}}{1-(-3)^{-1}}\right]$ $T_2 = 2\left[\frac{1+(-3)^{-2}}{1-(-3)^{-2}}\right]$	Mostly did well and a handful of students did not keep to the format for recursive method of induction. A few students had problem with working with base -3.

$$T_3 = 2\left[\frac{1+(-3)^{-3}}{1-(-3)^{-3}}\right]$$

$$\vdots \qquad \vdots$$

$$T_{k-1} = 2\left[\frac{1+(-3)^{-(k-1)}}{1-(-3)^{-(k-1)}}\right]$$

$$T_k = 2\left[\frac{1+(-3)^{-k}}{1-(-3)^{-k}}\right] \qquad ①$$

Required to prove $T_{k+1} = 2\left[\frac{1+(-3)^{-(k+1)}}{1-(-3)^{-(k+1)}}\right]$

Proof: $\text{LHS} = T_{k+1}$

$$= \frac{4+T_k}{1+T_k} \quad \text{(from recursive definition)}$$

$$= \frac{4+2\left[\frac{1+(-3)^{-k}}{1-(-3)^{-k}}\right]}{1+2\left[\frac{1+(-3)^{-k}}{1-(-3)^{-k}}\right]} \quad \text{(from assumption)} \quad ①$$

$$= \frac{4\left[1-(-3)^{-k}\right]+2\left[1+(-3)^{-k}\right]}{1\left[1-(-3)^{-k}\right]+2\left[1+(-3)^{-k}\right]}$$

$$= \frac{4.3^k-4(-1)^k+2.3^k+2(-1)^k}{3^k-(-1)^k+2.3^k+2(-1)^k}$$

$$= \frac{6.3^k-2(-1)^k}{3.3^k+(-1)^k}$$

$$= \frac{2\left[3.3^k-(-1)^k\right]}{3.3^k+(-1)^k}$$

$$= \frac{2\left[\frac{3.3^k-(-1)^k}{3.3^k}\right]}{\frac{3.3^k+(-1)^k}{3.3^k}}$$

$$= \frac{2\left[\frac{3.3^k}{3.3^k}-\frac{(-1)^k}{3.3^k}\right]}{\frac{3.3^k}{3.3^k}+\frac{(-1)^k}{3.3^k}}$$

	$= \frac{2\left[1-\frac{(-1)^k}{3^{k+1}}\right]}{1+\frac{(-1)^k}{3^{k+1}}}$ $= \frac{2\left[1-\frac{(-1)^k}{3^{k+1}}\times\frac{-1}{-1}\right]}{1+\frac{(-1)^k}{3^{k+1}}\times\frac{-1}{-1}}$ $= \frac{2\left[1+\frac{(-1)^{k+1}}{3^{k+1}}\times\frac{1}{1}\right]}{1-\frac{(-1)^{k+1}}{3^{k+1}}\times\frac{1}{1}}$ $= \frac{2\left[1+\left(-\frac{1}{3}\right)^{k+1}\right]}{\left[1-\left(-\frac{1}{3}\right)^{k+1}\right]}$ $= \frac{2\left[1+(-3)^{-(k+1)}\right]}{\left[1-(-3)^{-(k+1)}\right]}$ (closed definition) $\therefore$ LHS = RHS If true for $n = k$, hence proven true for $n = k + 1$. Since true for T_1, hence proven true for $T_{1+1} = T_2$, $T_{2+1} = T_3$, and so on. Hence true for all positive integers $n \geq 1$. ①	
16.	(a)(ii) $\lim_{n\to\infty}\left(-\frac{1}{3}\right)^n = 0$ $\therefore \lim_{n\to\infty} T_n = \lim_{n\to\infty}\frac{2\left[1+\left(-\frac{1}{3}\right)^n\right]}{\left[1-\left(-\frac{1}{3}\right)^n\right]}$ $= \frac{2[1+0]}{[1-0]}$ $= 2$ ①	Mostly did well.
16.	(b)(i) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \; dx$ i.e. $I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^{n+2} x \; dx$	Mostly did well. Some students did not know their trig. Identity well enough that they missed the fact that derivative of tan x is $\sec^2 x$ and

	$\therefore I_{n+2}=\int_0^{\frac{\pi}{4}}\tan^{n+2}x\ dx$ $=\int_0^{\frac{\pi}{4}}\tan^n x\tan^2 x\ dx$ ① or other method $=\int_0^{\frac{\pi}{4}}\tan^n x\left(\sec^2 x-1\right)\ dx$ $=\int_0^{\frac{\pi}{4}}\tan^n x\sec^2 x-\tan^n x\ dx$ $=\int_0^{\frac{\pi}{4}}\tan^n x\sec^2 x\ dx-\int_0^{\frac{\pi}{4}}\tan^n x\ dx$ $I_{n+2}=\left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\frac{\pi}{4}}-I_n$ $I_{n+2}+I_n=\left[\frac{\tan^{n+1}\left(\frac{\pi}{4}\right)}{n+1}-\frac{\tan^{n+1}(0)}{n+1}\right]$ $I_{n+2}+I_n=\left[\frac{(1)^{n+1}}{n+1}-(0)\right]$ $\therefore I_{n+2}+I_n=\frac{1}{n+1}$ (as required) ①	that $\tan^2 x=\sec^2 x-1$, instead they were getting very lost by using integration by parts. 2 students did not answer the required expression of $I_{n+2}+I_n=\frac{1}{n+1}$
16.	(b)(ii) $I_4=\int_0^{\frac{\pi}{4}}\tan^4 x\ dx$ $I_4=\frac{1}{(2)+1}-I_2$ $I_2=\frac{1}{(0)+1}-I_0$ $I_0=\int_0^{\frac{\pi}{4}}\tan^0 x\ dx$ $=\int_0^{\frac{\pi}{4}}1\ dx$ $=[x]_0^{\frac{\pi}{4}}$ $=\frac{\pi}{4}$ ① $\therefore I_2=1-\frac{\pi}{4}$ $\therefore I_2=\frac{4-\pi}{4}$	A few students had problems with trig. identity $\tan^2 x+1=\sec^2 x$ A few students made the mistakes in identifying n value for I_4 as 4, rather than 2, hence had $\frac{1}{n+1}$ as $\frac{1}{(4)+1}$ instead of $\frac{1}{(2)+1}$.

	$\therefore I_4=\frac{1}{3}-\frac{4-\pi}{4}$ or $\frac{1}{3}-\left(1-\frac{\pi}{4}\right)$ $=\frac{4-3(4-\pi)}{12}$ or $\frac{1}{3}-1+\frac{\pi}{4}$ $\therefore I_4=\frac{3\pi-8}{12}$ or $\frac{\pi}{4}-\frac{2}{3}$ ①	
16.	(b)(iii) $J_n=(-1)^n I_{2n}$ $J_{n-1}=(-1)^{n-1} I_{2(n-1)}$ $J_n-J_{n-1}=(-1)^n I_{2n}-(-1)^{n-1} I_{2(n-1)}$ for $n\geq 1$ $=(-1)^{n-1}\left[(-1)I_{2n}-I_{2(n-1)}\right]$ $=(-1)^{n-1}(-1)\left[I_{2n}+I_{2(n-1)}\right]$ $=(-1)^n\left[I_{2n}+I_{2(n-1)}\right]$ and from part (i) $I_{n+2}+I_n=\frac{1}{n+1}$ $I_{(2n)+2}+I_{(2n)}=\frac{1}{(2n)+1}$ $\therefore I_{(2n)}+I_{(2n)-2}=\frac{1}{(2n-2)+1}$ $\therefore J_n-J_{n-1}=(-1)^n\left(\frac{1}{2n-1}\right)$ for $n\geq 1$	**“Show”** question: a few students skipped steps.
16.	(b)(iv) From part (iii) $J_n-J_{n-1}=(-1)^n\left(\frac{1}{2n-1}\right)$ $\therefore J_m=(J_m-J_{m-1})+(J_{m-1}-J_{m-2})+(J_{m-2}-J_{m-3})$ $+\ldots+(J_1-J_0)+J_0$ $=J_0+(J_m-J_{m-1})+(J_{m-1}-J_{m-2})+(J_{m-2}-J_{m-3})$ $+\ldots+(J_1-J_0)$ $=J_0+(J_m-J_{m-1})+(J_{m-1}-J_{m-2})+(J_{m-2}-J_{m-3})$ $=J_0+(-1)^1\left(\frac{1}{2(1)-1}\right)+(-1)^2\left(\frac{1}{2(2)-1}\right)$ $+(-1)^3\left(\frac{1}{2(3)-1}\right)+\ldots+(-1)^m\left(\frac{1}{2m-1}\right)$ ① $=(-1)^0 I_{2(0)}+\sum_{n=1}^{m}(-1)^n\left(\frac{1}{2n-1}\right)$ ① $=I_0+\sum_{n=1}^{m}\frac{(-1)^n}{2n-1}$	Mostly did well. Easy question to show since the series for J_m. Students need to show the substitution of $J_n-J_{n-1}=(-1)^n\left(\frac{1}{2n-1}\right)$ for the sigma notation, and either show how $J_0=\frac{\pi}{4}$.

	$\therefore J_m = \frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$ (as required)	
16.	(b)(v) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \; dx$ Let $u = \tan x$ $\frac{du}{dx} = \sec^2 x$ $\frac{du}{dx} = \tan^2 x + 1$ $\frac{dx}{du} = \frac{1}{\tan^2 x + 1}$ $dx = \frac{1}{u^2 + 1} du$ When $x = \frac{\pi}{4}, \; u = 1$ $x = 0, \; u = 0$ $\therefore I_n = \int_0^1 u^n \; \frac{du}{1+u^2}$ $\therefore I_n = \int_0^1 \frac{u^n}{1+u^2} \; du$ (as required) ①	Mostly did well.
16.	(b)(vi) From (i) $I_n + I_{n+2} = \frac{1}{n+1}$. Since $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \; dx > 0$, and $I_{n+2} = \int_0^{\frac{\pi}{4}} \tan^{n+2} x \; dx > 0$ Then $I_n < \frac{1}{n+1}$ and $I_{n+2} < \frac{1}{n+1}$ Hence $0 < I_n < \frac{1}{n+1}$. ① As $\lim_{n\to\infty} \frac{1}{n+1} = 0$ i.e. $\lim_{n\to\infty} I_n = 0$ and $\lim_{n\to\infty} I_{2n} = 0$ and $\lim_{n\to\infty} J_n = \lim_{n\to\infty} (-1)^n I_{2n}$ ① $\therefore \lim_{n\to\infty} J_n = 0$ (and $\frac{\pi}{4} + \sum_{n=1}^{m} \frac{(-1)^n}{2n-1} = 0$ $\frac{\pi}{4} = -\sum_{n=1}^{m} \frac{(-1)^n}{2n-1}$	Some students have approached from different perspective and did well. Students that did not give a comprehensive approach for either or both deductions were not awarded the marks.

	$\frac{\pi}{4}=\sum_{n=1}^{m}\frac{\left(-1\right)^{n+1}}{2n-1}\Bigg)$	